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Integrability of maximal functions in Orlicz spaces of variable exponent

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1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensinonal Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r. For a locally integrable function f on $\mathbf{R}^n$, we consider the maximal function Mf defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy,$$

where $|B(x,r)|$ denotes the volume of $B(x,r)$.

In classical (constant exponent) Lebesgue spaces, we know the following basic facts about the maximal operator (see the book by Stein [29, Chapter 1]):

(i) If $q>1$, then

$$\|Mf\|_q\leq C\|f\|_q \qquad \text{for all } f\in L^q(\Omega).$$

(ii) If Ω is bounded, then

$$\|Mf\|_1\leq C\|f\|_{L\log L} \qquad \text{for all } f\in L\log L(\Omega).$$

Following Orlicz [25] and Kováčik and Rákosník [21], we consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ and the space of all measurable functions f on $\mathbf{R}^n$ satisfying

$$\int\left|\frac{f(y)}{\lambda}\right|^{p(y)}dy<\infty$$

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for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}.$$

In connection with these classical results, a natural question arises about conditions on $p(\cdot)$ implying the inequality

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}$$

for $f \in L^{p(\cdot)}(\Omega)$. Diening [6] is the first who treated the local boundedness of the maximal operator, and Cruz-Uribe, Fiorenza and Neugebauer [5] showed that this remains true for $\mathbf{R}^n$ when $p(\cdot)$ satisfies a log-Hölder condition on $\mathbf{R}^n$ including the point at infinity. In fact, they showed the following result.

THEOREM A. Let Ω be an open set, and let $p(\cdot)$ be a variable exponent in Ω satisfying $1 < \inf_\Omega p(x) \le \sup_\Omega p(x) < \infty$,

$$|p(x) - p(y)| \le \frac{C}{\log(1/|x-y|)}, \qquad x,\ y \in \Omega,\ |x-y| < \frac{1}{2}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log|x|}, \qquad x,\ y \in \Omega,\ |y| > |x| > e.$$

Then the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, that is,

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)} \qquad \text{for all } f \in L^{p(\cdot)}(\Omega).$$

In this paper we aim to extend their results and the authors [10].

We say that a positive nondecreasing function φ on the interval $[0,\infty)$ satisfies (P) if there exist $\varepsilon_0 > 0$ and $0 < r_0 < 1/e$ such that

$$\text{(P)} \qquad (\log(1/r))^{-\varepsilon_0}\varphi(1/r) \qquad \text{is nondecreasing on } (0, r_0).$$

For positive nondecreasing functions φ and ψ satisfying (P), let us assume that our variable exponent $p(\cdot)$ is a positive continuous function on $\mathbf{R}^n$ satisfying :

(p1) $1 < p^- = \inf_{\mathbf{R}^n} p(x) \le \sup_{\mathbf{R}^n} p(x) = p^+ < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{\log\varphi(1/|x-y|)}{\log(1/|x-y|)}$ whenever $|x-y| < 1/e$;

(p3) $|p(x) - p(y)| \le \dfrac{\log\psi(|x|)}{\log|x|}$ whenever $|y| > |x|/2 > e/2$.

Condition (p3) implies that $p(\cdot)$ has a finite limit p_∞ at infinity and

$$\text{(p4)} \qquad |p(x) - p_\infty| \le \frac{\log \psi(|x|)}{\log |x|} \qquad \text{whenever } |x| > e.$$

If $f \in L^{p(\cdot)}(\mathbf{R}^n)$, then we find for $B_0 = B(x_0, r_0)$ with $0 < r_0 < 1/e$

$$\int_{B_0} |f(y)|^{p(x_0)} |f(y)|^{\frac{\log \varphi(1/|x_0-y|)}{\log(1/|x_0-y|)}} dy < \infty \Rightarrow \int_{B_0} |f(y)|^{p(y)} dy < \infty$$

$$\Rightarrow \int_{B_0} |f(y)|^{p(x_0)} |f(y)|^{\frac{-\log \varphi(1/|x_0-y|)}{\log(1/|x_0-y|)}} dy < \infty.$$

Since the left and right hand sides are considered to be Orlicz-type conditions, the class $L^{p(\cdot)}(\mathbf{R}^n)$ is related to certain Orlicz spaces. More precisely, see Remarks 2.9 – 2.11 below.

Now we set

$$\Phi_A(x,t) = t^{p(x)} \varphi(t)^{-A/p(x)},$$
$$\Psi_A(x,t) = t^{p(x)} \psi(t^{-1})^{-A/p(x)}$$

and

$$\mathcal{P}_A(x,t) = \min\{\Phi_A(x,t), \Psi_A(x,t)\}.$$

In view of Lemma 2.1 (ii) below, we see that $\Phi_A(x,\cdot)$, $\Psi_A(x,\cdot)$ and $\mathcal{P}_A(x,\cdot)$ are quasi-increasing on $(0,\infty)$; for example, there exists $C > 1$ such that

$$\Phi_A(x,s) \le C\Phi_A(x,t) \qquad \text{whenever } 0 < s < t \text{ and } x \in \mathbf{R}^n. \tag{1.1}$$

We define the quasi-norm

$$\|f\|_{\mathcal{P}_A(\cdot,\cdot)} = \inf\left\{\lambda > 0 : \int \mathcal{P}_A(x, |f(x)|/\lambda) dx \le 1\right\}$$

and denote by $L^{\mathcal{P}_A(\cdot,\cdot)}(\mathbf{R}^n)$ the family of all functions f on $\mathbf{R}^n$ such that $\|f\|_{\mathcal{P}_A(\cdot,\cdot)} < \infty$.

It is well known (see for example Cianchi [3]) that the maximal operator is bounded in the Orlicz space consisting of functions f satisfying

$$\int_{\mathbf{R}^n} \Phi(|f(y)|) dy < \infty,$$

where Φ is a convex function on the interval $[0,\infty)$ such that $\Phi(r)/r^p$ is nondecreasing for some $p > 1$. As an extension of this fact to the variable exponent case, we first aim to establish the following result concerning the boundedness of maximal operators.

THEOREM 1.1 *The maximal operator M is bounded from $L^{p(\cdot)}(\mathbf{R}^n)$ to $L^{\mathcal{P}_A(\cdot,\cdot)}(\mathbf{R}^n)$ when $A > n$.*

If φ and ψ are constants, then we can take $A = 0$. Hence our theorem extends the results by D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer [5]. In Theorem 1.1, we can not take $A < n$ in general, as will be seen from Remark 2.11 below.

In his paper [12], P. Hästö studied local integrability of maximal functions for the exponent

$$p(x) = 1 + a\frac{\log(e + \log(e + \delta_K(x)^{-1}))}{\log(e + \delta_K(x)^{-1})},$$

where $\delta_K(x)$ denotes the distance of x from the compact set K in $\mathbf{R}^n$. Further, P. Harjulehto and P. Hästö [13] showed continuity of Sobolev functions for exponents of the form

$$p(x) = p_0 + a\frac{\log(e + \log(e + \delta_K(x)^{-1}))}{\log(e + \delta_K(x)^{-1})},$$

which can be seen as an extension of the fact : if $u \in W^{1,n}_{loc}(\mathbf{R}^n)$ satisfies

$$\int_{\mathbf{R}^n} |\nabla u(x)|^n (\log(1 + |\nabla u(x)|))^a dx < \infty$$

with $a > n - 1$, then u is continuous on $\mathbf{R}^n$. For further related results, see [9] and [23].

If G is a bounded open set in $\mathbf{R}^n$, then the conclusion of our theorem implies

$$\int_G |Mf(x)|^{p(x)} \varphi(Mf(x))^{-A/p(x)} dx < \infty$$

for $f \in L^{p(\cdot)}(\mathbf{R}^n)$, which gives the Orlicz-type condition

$$\int_{B(x_0,r_0)} |Mf(x)|^{p(x_0)} \{|Mf(x)|^{-\frac{\log \varphi(1/|x_0-x|)}{\log(1/|x_0-x|)}} \varphi(Mf(x))^{-A/p(x_0)}\} dx < \infty$$

for small r_0.

To show Theorem 1.1, different from the bounded domain case, we need to discuss a boundedness property for the Hardy operator defined by

$$Hf(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)| dy.$$

As applications of Theorem 1.1, we discuss Sobolev's type inequality for Riesz potentials of functions in Orlicz spaces of variable exponent by use of the so called Hedberg trick (see [19]). For the case of variable exponents satisfying the so called log-Hölder condition, there are many papers, e.g, Almeida-Samko [1], Capone-Cruz-Uribe-Fiorenza [2], Cruz-Uribe-Fiorenza-Martell-Pérez [4], Diening [7], Edmunds-Rákosník [8], Futamura-Mizuta [9], Futamura-Mizuta-Shimomura [10, 11], Mizuta-Shimomura [24], Harjulehto-Hästö [13], Harjulehto-Hästö-Koskenoja [14, 15], Harjulehto-Hästö-Koskenoja-Varonen [16], Harjulehto-Hästö-Latvala [17], Harjulehto-Hästö-Pere [18], Kokilashvili-Samko [20], Samko-Shargorodsky-Vakulov [27] and Samko-Vakulov [28].

2 Proof of Theorem 1.1

Throughout this paper, let C denote various constants independent of the variables in question.

First we note the following result, which can be derived by condition (P).

LEMMA 2.1 ([22], [23, Lemma 2.1]).

(i) *$\varphi(r)$ is of log-type, that is, there exists $C > 0$ such that*

$$C^{-1}\varphi(r) \le \varphi(r^2) \le C\varphi(r) \qquad \text{whenever } r > 0.$$

(ii) *For $\delta > 0$, $r^{-\delta}\varphi(r)$ is almost decreasing, that is, there exists $C > 0$ such that*

$$r_2^{-\delta}\varphi(r_2) \le C r_1^{-\delta}\varphi(r_1) \qquad \text{whenever } r_2 > r_1 > 0.$$

(iii) *There exists $0 < r_0 < 1/e$ such that $\omega_1(r) = \log\varphi(1/r)/\log(1/r)$ is nondecreasing on $(0, r_0]$; set $\omega_1(r) = \omega_1(r_0)$ for $r > r_0$.*

(iv) *There exists $R_0 > e$ such that $\omega_2(r) = \log\psi(r)/\log r$ is nonincreasing on $[R_0, \infty)$; set $\omega_2(r) = \omega_2(R_0)$ for $0 < r < R_0$.*

In view of (i) we see that

(i′) for each $\gamma > 0$ there exists $C > 0$ such that

$$C^{-1}\varphi(r) \le \varphi(r^\gamma) \le C\varphi(r) \qquad \text{whenever } r > 0.$$

Recall

$$\Phi_A(x,t) = t^{p(x)}\varphi(t)^{-A/p(x)}$$

for $A > n$. Setting

$$\|f\|_{\Phi_A(\cdot,\cdot)} = \inf\left\{\lambda > 0 : \int \Phi_A(x, |f(x)|/\lambda)dx \le 1\right\},$$

we denote by $L^{\Phi_A(\cdot,\cdot)}(\mathbf{R}^n)$ the family of all functions f on $\mathbf{R}^n$ such that $\|f\|_{\Phi_A(\cdot,\cdot)} < \infty$. Then we see that $\|\cdot\|_{\Phi_A(\cdot,\cdot)}$ is a quasi-norm, that is,

(i) $\|f\|_{\Phi_A(\cdot,\cdot)} = 0$ if and only if $f = 0$,

(ii) $\|kf\|_{\Phi_A(\cdot,\cdot)} = |k|\|f\|_{\Phi_A(\cdot,\cdot)}$,

(iii) $\|f + g\|_{\Phi_A(\cdot,\cdot)} \le C\left(\|f\|_{\Phi_A(\cdot,\cdot)} + \|g\|_{\Phi_A(\cdot,\cdot)}\right)$

for $f, g \in L^{\Phi_A(\cdot,\cdot)}(\mathbf{R}^n)$ and a real number k. The same is true for $\|\cdot\|_{\Psi_A(\cdot,\cdot)}$ as well as $\|\cdot\|_{\Phi_A(\cdot,\cdot)}$.

EXAMPLE 2.2 (1) Our typical example of φ is

$$\varphi(r) = a(\log r)^b(\log(\log r))^c \qquad \text{for } r \geq R_0$$

and $\varphi(r) = \varphi(R_0)$ for $0 \leq r < R_0$ if the numbers $R_0 > e$, $a > 0$, $b \geq 0$ and c are chosen so that $\varphi(r)$ is nondecreasing on $(0, \infty)$.

(2) For a positive nondecreasing function φ satisfying (P), set

$$\omega(r) = \frac{\log \varphi(1/r)}{\log(1/r)} \qquad (0 < r \leq r_0 = 1/R_0).$$

Then we see that

$$|\omega(s) - \omega(t)| \leq \omega(|s - t|) \qquad \text{for all } 0 < s, t \leq r_0.$$

For this, we have only to see that

$$\omega(s+t) \;\leq\; \log \varphi(1/(s+t)) \left\{ \frac{1}{\log(1/s)} + \frac{1}{\log(1/t)} \right\} \leq \omega(s) + \omega(t)$$

for $s, t > 0$ with $s + t \leq r_0$.

(3) Let K be a compact set in $\mathbf{R}^n$ and denote the distance of x from K by $\delta_K(x)$. For φ as in the introduction and $p_0 > 1$,

$$p(x) = p_0 + \frac{\log \varphi(1/\delta_K(x))}{\log(1/\delta_K(x))} \qquad \text{for } x \text{ near } K$$

can be extended to an exponent satisfying conditions (p1) and (p2).

(4) For $p_0 > 1$ and $\delta > 0$,

$$p(x) = p_0 + \left(\frac{1}{\log(e + \log(e + |x|))} \right)^{\delta}$$

satisfies (p1) – (p4) with φ and ψ replaced by suitable constants.

For a proof of Theorem 1.1, we need the following result. For this purpose, it is worth to see that

$$(\omega_1) \qquad r^{-\omega_1(r)} \leq C\varphi(1/r)$$

and

$$(\omega_2) \qquad r^{\omega_2(r)} \leq C\psi(r)$$

whenever $r > 0$.

LEMMA 2.3 *Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{p(\cdot)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$. Set*

$$F = F(x, r, f) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy$$

and

$$G = G(x, r, f) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)}dy.$$

Then

$$F \le CG^{1/p(x)} \varphi(G)^{n/p(x)^2}.$$

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{p(\cdot)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$. First consider the case when $G \ge 1$. Note by (ω_1) that

$$G^{\omega_1(CG^{-1/n})} \le C\varphi(G)^n$$

and

$$\varphi(G)^{\omega_1(CG^{-1/n})} \le C.$$

Since $\|f\|_{p(\cdot)} \le 1$ by our assumption, we find

$$\int f(y)^{p(y)}dy \le 1,$$

so that $G \le 1/|B(x,r)|$. Hence we have for $y \in B(x,r)$,

$$\begin{aligned} \left\{G^{1/p(x)}\varphi(G)^{n/p(x)^2}\right\}^{-p(y)} &\le \left\{CG^{1/p(x)}\varphi(G)^{n/p(x)^2}\right\}^{-p(x)+\omega_1(r)} \\ &\le \left\{CG^{1/p(x)}\varphi(G)^{n/p(x)^2}\right\}^{-p(x)+\omega_1(CG^{-1/n})} \le CG^{-1}, \end{aligned}$$

so that

$$\begin{aligned} F &\le G^{1/p(x)}\varphi(G)^{n/p(x)^2} + \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \left\{\frac{f(y)}{G^{1/p(x)}\varphi(G)^{n/p(x)^2}}\right\}^{p(y)-1} dy \\ &\le CG^{1/p(x)}\varphi(G)^{n/p(x)^2}. \end{aligned}$$

In the case $G \le 1$, noting that $f(y) \le f(y)^{p(y)}$ for $y \in \mathbf{R}^n$, we find

$$F \le G \le CG^{1/p(x)} \le CG^{1/p(x)}\varphi(G)^{n/p(x)^2}$$

since $\varphi(0) > 0$. Now the result follows. □

PROPOSITION 2.4 *Let $0 < R < \infty$. Then the maximal operator M is bounded from $L^{p(\cdot)}(B(0,R))$ to $L^{\Phi_A(\cdot,\cdot)}(\mathbf{R}^n)$ when $A > n$.*

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{p(\cdot)} \le 1$ such that $f = 0$ outside $B(0, R)$. We write

$$f = f\chi_{\{y: f(y) \ge 1\}} + f\chi_{\{y: f(y) < 1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E.

Now take p_0 such that $1 < p_0 < p^-$, and set $p_0(x) = p(x)/p_0$. Then we see that

$$\int_{B(0,R)} f_1(y)^{p_0(y)} dy \le \int_{B(0,R)} f(y)^{p(y)} dy \le 1,$$

so that $\|f_1\|_{p_0(\cdot)} \le 1$. Applying Lemma 2.3 with $p(x)$ and $\varphi(r)$ replaced by $p_0(x)$ and $\varphi(r)^{1/p_0}$ respectively, we find

$$Mf_1(x) \le C\{Mg_0(x)\}^{1/p_0(x)} \varphi(Mg_0(x))^{n/\{p_0 p_0(x)^2\}}$$

for $x \in B(0, 2R)$, where $g_0(y) = f(y)^{p_0(y)}$. Since $Mf_2(x) \le 1$, we establish

$$Mf(x) \le C\{Mg_0(x)\}^{1/p_0(x)} \varphi(Mg_0(x))^{n/\{p_0 p_0(x)^2\}} + C,$$

so that Lemma 2.1 gives

$$\{Mf(x)\}^{p(x)} \varphi(Mf(x))^{-np_0/p(x)} \le C\left(Mg_0(x) + 1\right)^{p_0}.$$

Thus it follows that

$$\Phi_A(x, Mf(x)) \le C + C\{Mg_0(x)\}^{p_0}$$

with $A = np_0$. Hence, by the well-known boundedness of the maximal operator, we insist that

$$\int_{B(0,2R)} \Phi_A(x, Mf(x))dx \le C.$$

If $|x| \ge 2R$, then

$$Mf(x) \le C|x|^{-n} \int_{B(0,R)} \{1 + f(y)^{p(y)}\} dy \le C|x|^{-n},$$

which proves

$$\int_{\mathbf{R}^n \setminus B(0,2R)} \Phi_A(x, Mf(x))dx \le C.$$

Thus the required result is proved. □

LEMMA 2.5 *Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ on $B(0, R_0)$ and $f < 1$ on $\mathbf{R}^n$. Then*

$$F \le C\{G\psi(G^{-1})^{n/p(x)}\}^{1/p(x)} + C\gamma(x) + CHf(x)$$

whenever $|x| \ge e$, where $\gamma(x) = |x|^{-n/p(x)} \psi(|x|)^{n/p_\infty^2}$ and

$$Hf(x) = \frac{1}{|B(0, |x|)|} \int_{B(0,|x|)} |f(y)| dy.$$

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ on $B(0, R_0)$ and $f < 1$ on $\mathbf{R}^n$. Then note that

$$G = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy < 1.$$

Let $|x| \geq e$. In the case $G \geq |x|^{-n}$, we have by (p3) and (ω_2)

$$\begin{aligned}\left\{G^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}\right\}^{-p(y)} &\leq \left\{CG^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}\right\}^{-p(x)-\omega_2(|x|)} \\ &\leq \left\{CG^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}\right\}^{-p(x)-\omega_2(G^{-1/n})} \\ &\leq CG^{-1}\end{aligned}$$

for $|y| > |x|/2$. Hence we find

$$\begin{aligned}&\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)dy \\ \leq\ & G^{1/p(x)}\psi(G^{-1})^{n/p(x)^2} \\ &+\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)\left\{\frac{f(y)}{G^{1/p(x)}\psi(G^{-1})^{n/p(x)^2}}\right\}^{p(y)-1} dy \\ \leq\ & CG^{1/p(x)}\psi(G^{-1})^{n/p(x)^2} \\ \leq\ & CG^{1/p(x)}\psi(G^{-1})^{n/p_\infty^2}.\end{aligned}$$

In the case $G \leq |x|^{-n}$, we see that

$$\begin{aligned}&\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)dy \\ \leq\ & |x|^{-n/p(x)}\psi(|x|)^{n/p(x)^2} \\ &+\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)\left\{\frac{f(y)}{|x|^{-n/p(x)}\psi(|x|)^{n/p(x)^2}}\right\}^{p(y)-1} dy \\ \leq\ & C|x|^{-n/p(x)}\psi(|x|)^{n/p(x)^2} \leq C\gamma(x).\end{aligned}$$

Finally we obtain

$$\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)} f(y)dy \leq CHf(x),$$

which completes the proof. □

LEMMA 2.6 *Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ on $B(0, R_0)$, $f < 1$ on $\mathbf{R}^n$ and*

$$G_0 = \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(y)^{p(y)}dy \leq C|x|^{-\delta} \tag{2.1}$$

for some $C > 0$ and $\delta > 0$ independent of x and f. If $0 < \beta < n$, then

$$Hf(x) \le C\left\{G_0\psi(G_0^{-1})^{\beta/p(x)}\right\}^{1/p(x)} + C|x|^{-\beta/p(x)}$$

for $|x| \ge R_0$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying $f = 0$ on $B(0, R_0)$, $f < 1$ on $\mathbf{R}^n$ and (2.1). For $|x| \ge R_0$, we have by Hölder's inequality

$$\begin{aligned} Hf(x)^{p(x)} &\le \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(y)^{p(x)}dy \\ &= \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)\cap E} f(y)^{p(x)}dy + \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)\setminus E} f(y)^{p(x)}dy \\ &= H_1 + H_2, \end{aligned}$$

where $E = \{y \in \mathbf{R}^n \setminus B(0, R_0) : |y|^{-\beta/p(x)} \le f(y) < 1\}$. Note that

$$H_2 \le C|x|^{-\beta}.$$

If $y \in B(0,|x|) \cap E$, then

$$f(y)^{p(x)} \le f(y)^{p(y)-\omega_2(|y|)} \le f(y)^{p(y)}\psi(|y|)^{\beta/p(x)} \le f(y)^{p(y)}\psi(|x|)^{\beta/p(x)},$$

so that

$$H_1 \le \psi(|x|)^{\beta/p(x)}G_0,$$

which together with (2.1) gives

$$H_1 \le C\psi(G_0^{-1})^{\beta/p(x)}G_0,$$

as required. □

Applying Hardy's inequality, we can prove the following result.

LEMMA 2.7 *For $1 < p_0 < \infty$,*

$$\|Hg_0\|_{p_0} \le C\|g_0\|_{p_0}$$

for all functions $g_0 \in L^{p_0}(\mathbf{R}^n)$.

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{p(\cdot)} \le 1$. Write

$$f = f\chi_{\{y:f(y)\ge 1\}} + f\chi_{\{y:f(y)<1\}} = f_1 + f_2.$$

We have by Lemma 2.3,

$$Mf_1(x) \le C\{Mg(x)\}^{1/p(x)}\varphi(Mg(x))^{n/p(x)^2},$$

where $g(y) = f(y)^{p(y)}$, so that

$$\Phi_n(x, Mf_1(x)) \le CMg(x). \tag{2.2}$$

Hence, in view of the proof of Proposition 2.4, we see that

$$\int_{\mathbf{R}^n} \Phi_A(x, Mf_1(x))dx \le C$$

when $A > n$. Since $Mf_2 \le 1$ on $\mathbf{R}^n$, we have

$$\int_{B(0,e)} \Phi_A(x, Mf_2(x))dx \le C.$$

Further we find by Proposition 2.4

$$\int_{\mathbf{R}^n} \Phi_A(x, Mf_2'(x))dx \le C,$$

where $f_2'(y) = f_2(y)\chi_{B(0,e)}(y)$. Therefore it suffices to prove

$$\int_{\mathbf{R}^n\setminus B(0,e)} \Psi_A(x, Mf_2''(x))dx \le C, \tag{2.3}$$

where $f_2'' = f_2 - f_2'$.

Thus we may assume that $0 \le f < 1$ on $\mathbf{R}^n$ and $f = 0$ on $B(0,e)$. In this case, by Lemmas 2.5 and 2.6, we have for $0 < \beta < n$

$$\begin{aligned} Mf(x) &\le C\{Mg(x)\psi(Mg(x)^{-1})^{n/p(x)}\}^{1/p(x)} + C\gamma(x) + CHf(x) \\ &\le C\{Mg(x)\psi(Mg(x)^{-1})^{n/p(x)}\}^{1/p(x)} + C|x|^{-\beta/p(x)} \\ &\quad + C\{Hg(x)\psi(Hg(x)^{-1})^{n/p(x)}\}^{1/p(x)}, \end{aligned}$$

so that

$$\Psi_n(x, Mf(x)) \le CMg(x) + CHg(x) + C|x|^{-\beta} \tag{2.4}$$

for $|x| \ge e$. Let $1 < p_0 < p^-$. Applying (2.4) with $p(x)$ and $\psi(r)$ replaced by $p_0(x) = p(x)/p_0$ and $\psi(r)^{1/p_0}$ respectively, we find

$$\Psi_A(x, Mf(x))^{1/p_0} \le CMg_0(x) + CHg_0(x) + C|x|^{-\beta},$$

where $A = np_0$ and $g_0(y) = f(y)^{p_0(y)} = g(y)^{1/p_0}$. Hence, letting $\beta p_0 > n$, by Lemma 2.7 and the boundedness of maximal operator on L^{p_0}, we derive

$$\int_{\mathbf{R}^n\setminus B(0,e)} \Psi_A(x, Mf(x))dx \le C.$$

Thus the proof is completed. □

REMARK 2.8 In Theorem 1.1, we can replace $\mathcal{P}_A(x,t)$ by

$$\min\{t^{p(x)}\varphi(t)^{-A/p(x)}, t^{p(x)}\psi(t^{-1})^{-A/p_\infty}\}$$

or

$$\left[\min\{t\varphi(t)^{-A/p(x)^2}, t\psi(t^{-1})^{-A/p_\infty^2}\}\right]^{p(x)}.$$

REMARK 2.9 Let $p(\cdot)$ be the variable exponent such that

$$p(x) = p_0 + a\frac{\log\log(c_0/|x|)}{\log(c_0/|x|)}$$

for $x \in \mathbf{B} = B(0,1)$, where $a > 0$ and $c_0 > e$ are chosen so that $p(x) \geq p_0$ on $\mathbf{B}$ and $p(x)$ satisfies (p2) with $\varphi(r) = (\log(e+r))^a$. If f is a nonnegative measurable function in $L^{p(\cdot)}(\mathbf{B})$, then

$$\int_{\mathbf{B}} f(y)^{p_0}(\log(e+f(y)))^{an/p_0}dy < \infty.$$

In fact, letting $E = \{y \in \mathbf{B} : f(y) \leq |y|^{-n/p_0}(\log(e+|y|^{-1}))^{-\lambda}\}$ with $\lambda > (an/p_0 + 1)/p_0$, then

$$\begin{aligned}
&\int_{\mathbf{B}} f(y)^{p_0}(\log(e+f(y)))^{an/p_0}dy \\
\leq\ & C\int_E |y|^{-n}(\log(e+|y|^{-1}))^{an/p_0-\lambda p_0}dy + C\int_{\mathbf{B}\setminus E} f(y)^{p_0}(\log(e+f(y)))^{an/p_0}dy \\
\leq\ & C + C\int_{\mathbf{B}\setminus E} f(y)^{p(y)}dy < \infty.
\end{aligned}$$

REMARK 2.10 We next consider the converse part of Remark 2.10. Let $p(\cdot)$ be the variable exponent such that

$$p(x) = p_0 - a\frac{\log\log(c_0/|x|)}{\log(c_0/|x|)}$$

for $x \in \mathbf{B}$, where $a > 0$ and $c_0 > e$ are chosen so that $p(x) > 1$ on $\mathbf{B}$ and $p(x)$ satisfies (p2) with $\varphi(r) = (\log(e+r))^a$. If f is a nonnegative measurable function on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} f(y)^{p_0}(\log(e+f(y)))^{-an/p_0}dy < \infty,$$

then $f \in L^{p(\cdot)}(\mathbf{B})$.

REMARK 2.11 Consider the variable exponent

$$p(x) = \begin{cases} p_0 + a\frac{\log(e+\log(e+x_n^{-1}))}{\log(e+x_n^{-1})} & (x_n > 0) \\ p_0 & (x_n \leq 0) \end{cases}$$

for $x = (x_1, ..., x_n) \in \mathbf{B}$, where $a > 0$. Let

$$f(y) = \chi_{\mathbf{B}}(y) \times \begin{cases} |y|^{-n/p_0}(\log(e + |y|^{-1}))^{-1/p_0}(\log\log(e + |y|^{-1}))^{-\beta} & (y_n < 0) \\ 0 & (y_n \geq 0) \end{cases}$$

for $\beta p_0 > 1$. Then $f \in L^{p(\cdot)}(\mathbf{B})$. Noting that

$$Mf(x) \geq C|x|^{-n/p_0}(\log(e + |x|^{-1}))^{-1/p_0}(\log\log(e + |x|^{-1}))^{-\beta},$$

we have

$$\begin{aligned} &\int_{\mathbf{B}} Mf(x)^{p(x)}(\log(1 + Mf(x)))^{-K} dx \\ \geq\ & C \int_{\Gamma} |x|^{-n}(\log(e + |x|^{-1}))^{-1+an/p_0-K}(\log\log(e + |x|^{-1}))^{-\beta p_0} dx \end{aligned}$$

where $\Gamma = \{x = (x_1, ..., x_n) \in \mathbf{B} : x_n > |x|/2\}$. Hence

$$\int_{\mathbf{B}} Mf(x)^{p(x)}(\log(1 + Mf(x)))^{-K} dx = \infty$$

if $-K + an/p_0 > 0$. This implies that we can not take $A < n$ in Theorem 1.1, generally.

References

[1] A. Almeida and S. Samko, Characterization of Riesz and Bessel potentials on variable Lebesgue spaces, J. Funct. Spaces Appl. **4** (2006), 113–144.

[2] C. Capone, D. Cruz-Uribe and A. Fiorenza, The fractional maximal operator on variable L^p spaces, Rev. Mat. Iberoamericana **23** (2007), no. 3, 743-770.

[3] A. Cianchi, Strong and weak type inequalities for some classical operators in Orlicz spaces, J. London Math. Soc. **60** (1999), 187–202.

[4] D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez, The boundedness of classical operators on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239-264..

[5] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223-238, **29** (2004), 247-249.

[6] L. Diening, Maximal functions on generalized $L^{p(\cdot)}$ spaces, Math. Inequal. Appl. **7**(2) (2004), 245–253.

[7] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr. **263**(1) (2004), 31–43.

[8] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, II, Math. Nachr. **246-247** (2002), 53–67.

[9] T. Futamura and Y. Mizuta, Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent, Math. Inequal. Appl. **8** (2005), 619–631.

[10] T.Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for Riesz potential space of variable exponent, Math. Nachr. **279** (2006), 1463-1473.

[11] T. Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for variable exponent Riesz potentials on metric spaces, Ann. Acad. Sci. Fenn. Math. **31** (2006), 495–522.

[12] P. Hästö, The maximal operator in Lebesgue spaces with variable exponent near 1, Math. Nachr. **280** (2007), 74–82.

[13] P. Harjulehto and P. Hästö, A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces, Rev. Mat. Complut. **17** (2004), no. 1, 129–146.

[14] P. Harjulehto, P. Hästö and M. Koskenoja, Hardy's inequality in a variable exponent Sobolev space, Georgian Math. J. **12** (2005), no. 3, 431–442.

[15] P. Harjulehto, P. Hästö and M. Koskenoja, Properties of capacities in variable exponent Sobolev spaces, J. Anal. Appl. **5** (2007), no. 2, 71–92.

[16] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen, Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$, J. Funct. Spaces Appl. **1** (2003), no. 1, 17–33.

[17] P. Harjulehto, P. Hästö and V. Latvala, Sobolev embeddings in metric measure spaces with variable dimension, Math. Z. **254** (2006), no. 3, 591–609.

[18] P. Harjulehto, P. Hästö and M. Pere, Variable exponent Sobolev spaces on metric measure spaces, Funct. Approx. Comment. Math. **36** (2006), 79–94.

[19] L. I. Hedberg, On certain convolution inequalities, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[20] V. Kokilashvili and S. Samko, On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent, Z. Anal. Anwendungen. **22** (2003), 899-910.

[21] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618.

[22] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[23] Y. Mizuta, T. Ohno and T. Shimomura, Integrability of maximal functions for generalized Lebesgue spaces with variable exponent, Math. Nachr. **281** (2008), 386-395.

[24] Y. Mizuta and T. Shimomura, Sobolev's inequality for Riesz potentials with variable exponent satisfying a log-Hölder condition at infinity, J. Math. Anal. Appl. **311** (2005), 268-288.

[25] W. Orlicz, Über konjugierte Exponentenfolgen, Studia Math. **3** (1931), 200–211.

[26] M. Růžička, Electrorheological fluids : modeling and Mathematical theory, Lecture Notes in Math. **1748**, Springer, 2000.

[27] S. Samko, E. Shargorodsky and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. II, J. Math. Anal. Appl. **325** (2007), 745–751.

[28] S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, J. Math. Anal. Appl. **310** (2005), 229–246.

[29] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.